Random Matrices and L-functions

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Abstract

In recent years there has been a growing interest in connections between the statistical properties of number theoretical L-functions and random matrix theory. We review the history of these connections, some of the major achievements, and a number of applications.

1 The history in brief

Number theory and random matrix theory met, by chance, over a cup of tea in the common room at the Institute for Advanced Study in Princeton in the early 1970s. H.L. Montgomery, then a graduate student, had shown his latest work to F.J. Dyson. His main conclusion was a conjecture for the two-point correlation function of the zeros of the Riemann zeta function, which he had managed to prove for a limited range of correlations [57]. Dyson recognized this to be the same as the two-point correlation function, which he had calculated a decade earlier, for the eigenvalues of matrices drawn at random from $U(N)$ (the group of $N \times N$ unitary matrices) uniformly with respect to Haar measure [30].

The Riemann zeta function is extremely important in number theory because it allows for analytical techniques to be applied to the study of the distribution of prime numbers, and Montgomery's conjecture plays a central role in the theory of this distribution.

Although the full proof of Montgomery's conjecture has still escaped completion, Odlyzko's numerical computations have provided very strong evidence in support of it [59]. Working on the powerful computers of AT&T, in the 1980s Odlyzko calculated batches of the Riemann zeros high up on the critical line where the Riemann Hypothesis places them, and computed numerically the two-point correlation function, as well as many other statistics of the zeros. He also computed the distribution of the values of the zeta function. His numerics provide very convincing evidence that, as predicted by Montgomery's conjecture, the two-point correlation function of the Riemann zeros converges to the random matrix result of Dyson as zeros higher and higher on the critical line are considered. Odlyzko's numerical work continues today and he is currently working at the dizzying height of the 10^{22}nd Riemann zero.

In the 1980s the Riemann zeta function also took on a new life in mathematical and theoretical physics when it became a tool in the field of quantum chaos. The eigenvalues of complex quantum systems (eg. nuclei [4, 18, 35] and disordered mesoscopic systems [31]) and systems showing chaotic behaviour in the classical limit [12, 16, 17] display the statistics of the three standard ensembles of Hermitian matrices, the Gaussian Unitary Ensemble (GUE), the Gaussian Orthogonal Ensemble (GOE) and the Gaussian Symplectic Ensemble (GSE), or equivalently (in the appropriate limit) the three corresponding standard ensembles of unitary matrices: the CUE, COE and CSE. (Here the "C" stands for "circular".) However, while the eigenvalue statistics show certain behaviour which is easily identified and predicted by random matrices, there are further characteristics which are system-specific and are connected to the periodic orbits of the system under consideration [6]. These relate to the approach to the random-matrix limit as $\hbar \to 0$. It was realized by Berry [7] that very similar contributions result from the primes and behave in the same way with respect to the statistics of the Riemann zeros as do the periodic orbits to the statistics of the eigenvalues. In this case, the primes describe the approach to the random-matrix limit as the height up the critical line increases.

This launched an era of study of the Riemann zeta function in the field of quantum chaos [2, 6, 8, 9, 10, 11, 47]. Most of the results are reviewed in [11, 48]. In essence, the connection between the Riemann zeta function and the prime numbers was being used to point the way through more complicated periodic orbit calculations; and in return the familiarity which arose with the zeta function enabled physicists to contribute insights from physics to its study. In particular, it resulted in a return to the question of the statistics of the Riemann zeros when Bogomolny and Keating [13, 15] in 1995 and 1996 showed, subject to certain conjectures of Hardy and Littlewood concerning the distribution of primes, that not just the two-point, but the general n-point statistics of the Riemann zeros are the same as those of the eigenvalues of random unitary matrices in the limit as one looks at zeros infinitely high up the critical line.

At the same time, first Hejhal [37] with the three-point case, then Rudnick and Sarnak [61] generalized Montgomery's theorem by proving for a limited range of correlations, as in the two-point statistic, that the n-point correlations of the Riemann zeros high on the critical line coincide with the corresponding random unitary matrix statistics.

At the end of the 1990s, two developments occurred which illustrate how deeply random matrix theory is intertwined with the Riemann zeta function. Far from there being some accidental similarity between the zeros of this one function and the eigenvalues of random matrices, it became apparent that this connection was far more general. The zeta function is but one example of a broader class of functions known as L-functions. These all satisfy generalizations of the Riemann Hypothesis. For any individual L-function, it is believed that the zeros high up on the critical line are distributed like the eigenvalues of random unitary matrices, that is, exactly as in the case of the Riemann zeta function [57, 61]. More interesting, however, is the fact that it has been conjectured by Katz and Sarnak [45, 46] that averages over various families of L-functions, with the height up the critical line of each one fixed, are described not only by averages over the unitary group $U(N)$, but by averages over other classical compact groups, for example the orthogonal group $O(N)$ or the unitary symplectic group $USp(2N)$, depending upon the family in question. The eigenvalue statistics of these groups have also been found to occur in disordered superconducting systems [1].

The second of the two significant recent developments was the discovery that calculations purely within random matrix theory can suggest answers to important questions that number theorists have been unable using standard techniques to make progress on. In [51, 52] Keating and Snaith showed that by studying the value distribution and moments of the characteristic polynomial of a random matrix, one can make predictions about the value distribution and moments of the Riemann zeta function and other L-functions. The reason for this is clear: the characteristic polynomial of a random matrix has zeros (the eigenvalues of the matrix) which, conjecturally, show the same statistical behaviour of the zeros of L-functions. Thus random matrix theory can be put to very practical use in the study of L-functions.

Our aim in the following sections is to expand further on the developments outlined above. Specifically, we will concentrate on those aspects not covered in previous reviews [11, 48].

2 Pair correlations

We begin with some basic facts about the Riemann zeta function.

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1}$$

for $\mathrm{Re}s > 1$, where p labels the primes, and then by analytic continuation to the rest of the complex plane. It has a pole at $s = 1$, zeros at $s = -2, -4, -6, \ldots$ (the *trivial zeros*) and infinitely many zeros, called the *non-trivial zeros*, in the *critical strip* $0 < \mathrm{Re}s < 1$. The Riemann Hypothesis states that all of the non-trivial zeros lie on the *critical line* $\mathrm{Re}s = 1/2$; that is, $\zeta(1/2 + it) = 0$ has non-trivial solutions only when $t = t_n \in \mathbb{R}$ [63]. This is known to be true for at least 40% of the non-trivial zeros [20], for the first $1.5 \times 10^9 + 1$ of them [53], and for batches lying much higher [59]. (A distributed computing project claimed as of July 31st 2002 that the first $50,631,912,399$ non-trivial zeros lie on the line! [65]) The zeta function satisfies a *functional equation*:

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} \zeta(1-s). \tag{2}$$

In the following, for ease of presentation, we will assume the Riemann Hypothesis to be true, although this is not strictly necessary.

The mean density of the non-trivial zeros increases logarithmically with height t up the critical line. Specifically, defining *unfolded zeros* by

$$w_n = t_n \frac{1}{2\pi} \log \frac{t_n}{2\pi}, \tag{3}$$

it is known that

$$\lim_{W\to\infty} \frac{1}{W} \# \{w_n < W\} = 1. \tag{4}$$

The question then arises as to the statistical distribution of the unfolded zeros: are they equally spaced, with unit spacing between neighbours, randomly distributed with unit mean spacing, or do they have some other limiting distribution? Statistics such as the two-point correlation function contain information about this distribution. For example, let

$$F_\zeta(\alpha, \beta; W) = \frac{1}{W} \#\{w_n, w_m \in [0, W] : \alpha \le w_n - w_m < \beta\}; \tag{5}$$

that is, F_ζ measures correlations between pairs of unfolded zeros $w_n \in [0, W]$. The question is, first, does a limit distribution

$$F_\zeta(\alpha, \beta) = \lim_{W\to\infty} F_\zeta(\alpha, \beta; W) \tag{6}$$

exist, and second, if so, what can one say about it?

In 1973 Montgomery provided at least part of the answer to this. The two-point correlation function $R_{2,\zeta}(x)$ of the Riemann zeros is related to $F_\zeta(\alpha, \beta)$ by

$$F_\zeta(\alpha,\beta) = \int_\alpha^\beta (R_{2,\zeta}(x) + \delta(x))dx. \tag{7}$$

Using more general test functions, we can write

$$\begin{aligned} R_{2,\zeta}(f,W) &= \frac{1}{W} \sum_{\substack{j\neq k \\ w_j, w_k \leq W}} f(w_j - w_k) \\ &= \int_{-\infty}^{\infty} f(x) \frac{1}{W} \sum_{\substack{j\neq k \\ w_j, w_k \leq W}} \delta(x - w_j + w_k) dx. \end{aligned} \tag{8}$$

Montgomery's theorem is

Theorem 1 (Montgomery 1973 [57])

For test functions $f(x)$ such that

$$\hat{f}(\tau) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x \tau} dx$$

has support in $(-1,1)$, the following limit exists:

$$\lim_{W\to\infty} R_{2,\zeta}(f,W) = \int_{-\infty}^{\infty} f(x) R_2(x) dx,$$

with

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2.$$

Based on this result, Montgomery further conjectured that Theorem 1 is true for $\hat{f}(\tau)$ of unrestricted support. This we write as:

Conjecture 1 (Montgomery 1973 [57])

If, for $F_\zeta(\alpha,\beta)$ defined as in (6), we write

$$F_\zeta(\alpha,\beta) = \int_\alpha^\beta (R_{2,\zeta}(x) + \delta(x))dx,$$

then

$$R_{2,\zeta}(x) = R_2(x),$$

with $R_2(x)$ as in Theorem 1.

We are now in a position to explain the connection with random matrices. Random matrix theory was initiated by Eugene Wigner in the 1950s to describe the statistical distribution of nuclear energy levels. It was later developed by Dyson, Mehta, Gaudin and others in the 1960s into a rigorous area of mathematical physics. For a detailed introduction see [54]. One important result - the two-point correlation function for the eigenvalues of unitary matrices - was proved by Dyson in 1963. Let A be an $N \times N$ unitary matrix; that is, $A \in U(N)$. Denote the eigenvalues of

A by $\exp(i\theta_n)$, where $1 \le n \le N$ and $\theta_n \in \mathbb{R}$ (this follows from unitarity). Clearly the eigenphases θ_n have mean density $N/2\pi$, so the unfolded eigenphases

$$\phi_n = \theta_n \frac{N}{2\pi}, \tag{9}$$

have unit mean density (i.e. $\phi_n \in [0, N)$). Next let us define, by analogy with (5),

$$F(\alpha, \beta; A, N) = \frac{1}{N} \#\{\phi_n, \phi_m : \alpha \le \phi_n - \phi_m < \beta\}. \tag{10}$$

Now, the unitary group $U(N)$ comes with a natural invariant measure - Haar measure - which we will denote dA, and so one may compute the average over A of this function, with A taken uniformly with respect to Haar measure:

$$F_U(\alpha, \beta; N) = \int_{U(N)} F(\alpha, \beta; A, N) dA. \tag{11}$$

Dyson's theorem is then:

Theorem 2 (Dyson 1963 [30])
The limit distribution

$$F_U(\alpha, \beta) = \lim_{N \to \infty} F_U(\alpha, \beta; N),$$

(defined as in (10) and (11)) exists and takes the form

$$F_U(\alpha, \beta) = \int_\alpha^\beta [R_{2,U}(x) + \delta(x)]\, dx, \tag{12}$$

where $\delta(x)$ is Dirac's δ-function and

$$R_{2,U}(x) = R_2(x) \equiv 1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2.$$

The integrand in (12) may be thought of as the *two-point correlation function* for the eigenphases of a random unitary matrix, unfolded to have unit mean spacing. The fact that it is a non-trivial function of correlation distance x means that eigenphases are correlated in a non-trivial way. Note that in Theorem 2 the definition of $R_2(x)$ is identical to that in Theorem 1, so the Riemann zeros and the eigenvalues of matrices in $U(N)$ appear to have exactly the same non-trivial correlations in the limit of infinite height up the critical line on the one hand, and infinite matrix size on the other.

The basic idea underlying the method used to prove Montgomery's theorem is the following one. The formula (1) expressing $\zeta(s)$ as a product over the primes may be used to relate the zeros t_n to sums over the primes. Hence the pair correlation of the zeros may be written as a sum over pairs of primes p, q. The contribution from the *diagonal terms* with $p = q$ may be evaluated using the prime number theorem. The condition on $f(x)$ in Montgomery's theorem is designed so that only the diagonal terms contribute. Relaxing the condition on f would require the evaluation of the *off-diagonal terms* ($p \neq q$). The details of these calculations are reviewed in [48].

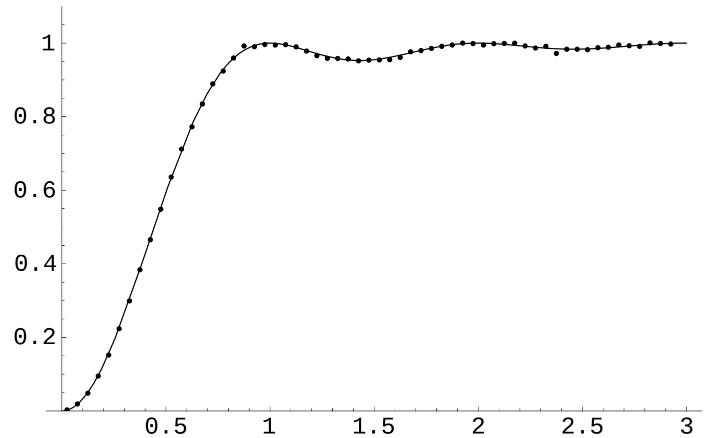

Figure 1: The two-point correlation function of 10^6 Riemann zeros around the height of the 10^{20}th zero (dots) and the two-point correlation function $R_2(x)$ (smooth curve), see Theorem 2, of the eigenvalues of matrices in $U(N)$ in the large-N limit. Figure courtesy of A. M. Odlyzko.

3 More on the zero statistics

There is substantial evidence in support of Montgomery's conjecture. Among other statistics, Odlyzko has computed the two-point correlation function numerically for batches of zeros high up on the critical line (e.g. near to the 10^{20}th zero) and his results [59] show striking agreement with $R_2(x)$ (given in Theorem 1), as illustrated by Figure 1.

Odlyzko's computations show a similarly convincing agreement for the distribution of spacings between adjacent unfolded Riemann zeros and the equivalent random matrix statistic [59]. This nearest-neighbour distribution depends on all of the n-point correlation functions and so suggests that Montgomery's conjecture generalizes to relate correlations between n-tuples of zeros and the corresponding correlations between n-tuples of eigenphases. That Theorem 1 also generalizes was proved by Hejhal for the three-point correlation function [37], and by Rudnick and Sarnak [61] for all the n-point correlations, as we will now explain.

Rudnick and Sarnak define an n-point correlation sum as follows. For a set B_N of N unfolded Riemann zeros $w_1 \leq w_2 \leq \cdots \leq w_N$ and for a test function f which satisfies

$$f(x) \equiv f(x_1, \ldots, x_n) \text{ is symmetric,} \tag{13a}$$

$$f(x + t(1, \ldots, 1)) = f(x) \text{ for } t \in \mathbb{R} \text{ (ie. } f(x) \text{ is a function of successive differences of the } x'\text{s), and} \tag{13b}$$

$$f(x) \to 0 \text{ rapidly as } |x| \to \infty \text{ in the hyperplane } \sum_j x_j = 0, \tag{13c}$$

define

$$R_{n,\zeta}(B_N, f) = \frac{n!}{N} \sum_{\substack{S \subset B_N \\ |S| = n}} f(S). \tag{14}$$

On the random matrix side, the n-point correlation function of the eigenphases of matrices from $U(N)$ is defined as

$$R_n(\theta_1, \ldots, \theta_n; N) = \frac{N!}{(N-n)!} \int_0^{2\pi} \cdots \int_0^{2\pi} P(\theta_1, \cdots \theta_N) d\theta_{n+1} \cdots d\theta_N, \tag{15}$$

where the *joint probability density function* of the eigenphases (derived from Haar measure) is given by

$$P(\theta_1, \ldots, \theta_N) = \frac{1}{N!(2\pi)^N} \prod_{1 \le m < n \le N} |e^{i\theta_n} - e^{i\theta_m}|^2; \tag{16}$$

that is, $P(\theta_1, \ldots, \theta_N) d\theta_1 \cdots d\theta_N$ is the probability that a matrix plucked from this ensemble has eigenphases between θ_1 and $\theta_1 + d\theta_1$, between θ_2 and $\theta_2 + d\theta_2$, and so on. These n-point correlation functions were evaluated by Dyson [30] and are given by

$$R_n(\theta_1, \ldots, \theta_n; N) = \det[K_N(\theta_j - \theta_k)]_{j,k=1,\ldots,n}, \tag{17}$$

where

$$K_N(\theta) = \frac{1}{2\pi} \frac{\sin(N\theta/2)}{\sin(\theta/2)}. \tag{18}$$

In the large-N limit, then, the n-point correlation function of the unfolded eigenphases is

$$R_n(\phi_1, \ldots, \phi_n) = \det[K(\phi_j - \phi_k)]_{j,k=1,\ldots,n}, \tag{19}$$

with $K(x) = \frac{\sin \pi x}{\pi x}$.

With these definitions, we have the theorem

Theorem 3 (Rudnick and Sarnak 1996 [61])

Let the test function f satisfy the conditions (13), and in addition assume that $\hat{f}(\xi)$ is supported in $\sum_j |\xi_j| < 2$. Further, assume the Riemann Hypothesis to be true. Then as $N \to \infty$,

$$R_{n,\zeta}(B_N, f) \to \int_{\mathbb{R}^n} f(x) R_n(x) \delta\left(\frac{x_1 + \cdots + x_n}{n}\right) dx_1 \cdots dx_n.$$

In the above theorem Rudnick and Sarnak make use of the assumption that the Riemann Hypothesis is true, but in [61] they define a smoothed version of the n-point correlation function and with this prove a similar result to Theorem 3 without it being necessary for the Riemann zeros to lie on the critical line. It is also important to note that Rudnick and Sarnak proved Theorem 3 not only for the zeros of the Riemann zeta function, but also for a whole class of other L-functions, as will be discussed in the following section.

The methods used in proving Theorem 3 are essentially a direct generalization of those underlying Theorem 1, described in outline at the end of Section 2; that is the result follows from an evaluation of the diagonal contributions to multiple sums over the primes.

In Theorem 3 as in the case of the two-point correlation function (Theorem 1), there is a restriction imposed on the support of the test function $f(x)$. No statistics of the Riemann zeros have been proved to agree with random matrix results outside of this range of correlation. To do so requires an evaluation of the off-diagonal contributions to multiple sums over the primes. However, there is a heuristic calculation of Bogomolny and Keating [13, 15] which shows the n-point correlation function to be, high on the critical line, exactly the same as that predicted by random matrix theory, without any restrictions on the correlation range. Their method uses a conjecture by

Hardy and Littlewood on the correlations between primes to evaluate the off-diagonal contributions needed. (See [48] for a review of the basic ideas in the case of the two-point correlation function.)

There have recently been other studies of the statistical distribution of the zeros of the Riemann zeta function. For example, the idea introduced in [52] that statistical properties of the zeta function at a finite height up the critical line might be modelled by finite-size random matrices, with height and size related by $N \sim \log t$, which will be discussed in later sections, has been verified in a systematic study of the fluctuations in the number of unfolded zeros lying in ranges of a given length (see the contribution by Coram and Diaconis [27] to this issue).

The conclusion to be drawn is that the statistical distribution of the Riemann zeros, in the limit as one looks infinitely high up the critical line, coincides with the statistical distribution of the eigenvalues of random unitary matrices, in the limit of large matrix size. (We note as well that a great deal is also known about the way in which zero statistics approach the large-height limit described by random matrix theory - see, for example, [8, 14, 49]. Results concerning the approach to this limit were reviewed recently by Berry and Keating in [11].)

4 Families of L-functions

As mentioned in the introduction, the connection between the Riemann zeros and random matrix theory is merely one example of a much more general relationship. The results of Rudnick and Sarnak hold not just for the Riemann zeta function but for other individual L-functions, as we will now discuss. Moreover, Katz and Sarnak [45, 46] have proposed a fundamental generalization - in terms of *families* of similar L-functions with each family subscribing to a symmetry type, not just the familiar unitary symmetry of $U(N)$, but also the symmetry corresponding to $O(N)$ and $USp(2N)$.

We will consider first the basic properties of L-functions, and then give a simple example of a family by way of illustration, before discussing further the results of Katz and Sarnak.

L-functions share the same general structure as the Riemann zeta function in that there exists for each a Dirichlet series and an Euler product over the primes p like those in (1), for example

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(\sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}} \right). \tag{20}$$

The coefficients a_n might be Dirichlet characters or Fourier coefficients of an automorphic cusp form. Whatever the source of the various L-functions, they have in common an analytic continuation beyond the region in which the series converges, and a functional equation which relates the L-function in one half of the complex plane with the other half. We will always consider L-functions to be normalized so that the line of symmetry is $\mathrm{Re} s = 1/2$. The Generalized Riemann Hypothesis then proposes that all the non-trivial zeros of a given L-function lie on this critical line. Theorem 3 of Rudnick and Sarnak applies equally well to any primitive L-function (one which does not factor into a product of L-functions) providing that the condition

$$\sum_p \frac{|a_{p^k} \log p|^2}{p^k} < \infty, \tag{21}$$

holds for any $k \geq 2$, where the sum is over the prime numbers (this clearly holds in the case of the Riemann zeta function). This suggests that for any individual L-function, the distribution of its zeros high on the critical line will display the same characteristics as the distribution of the eigenvalues of matrices pulled at random from $U(N)$ (uniformly with respect to Haar measure) for large N.

Now we turn to families of L-functions. Take, for example, an L-function with coefficients determined by a real Dirichlet character:

$$L(s,\chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s} = \prod_p \left[1 - \chi_d(p)p^{-s}\right]^{-1} \tag{22}$$

where $\chi_d(n) = \left(\frac{d}{n}\right)$ is Kronecker's extension of Legendre's symbol which is defined for p prime,

$$\left(\frac{d}{p}\right) = \begin{cases} +1 & \text{if } p \nmid d \text{ and } x^2 \equiv d (\text{mod} p) \text{ is soluble} \\ 0 & \text{if } p|d \\ -1 & \text{if } p \nmid d \text{ and } x^2 \equiv d (\text{mod} p) \text{ is not soluble} \end{cases}. \tag{23}$$

The character χ_d exists for all fundamental discriminants d, and the L-functions attached to these characters are said to form a family as we vary d. The family can be partially ordered by the *conductor* $|d|$. Katz and Sarnak noted that $U(N)$ and the other circular matrix ensembles which are standard, as mentioned in the introduction, in physics, COE= $U(N)/O(N)$ and CSE= $U(2N)/USp(2N)$, are only three of the dozen or so symmetric spaces characterized by Cartan. Amongst the others are the compact groups $USp(2N)$ and $O(N)$. In the context of physical systems, these other symmetric spaces were realized to be of importance independently by Altland and Zirnbauer [1]. From Katz and Sarnak we borrow Table 1 of the symmetric spaces which are important here.

Symmetry Type G	Realization of $G(N)$ as matrices
U, also called CUE	$U(N)$, the compact group of $N \times N$ unitary matrices.
SO (even)	$SO(2N)$, the compact group of $2N \times 2N$ unitary matrices A satisfying $A^tA = I$, $\det A = 1$.
SO (odd)	$SO(2N+1)$, the compact group of $(2N+1) \times (2N+1)$ unitary matrices A satisfying $A^tA = I$, $\det A = 1$.
Sp	$USp(2N)$, the compact group of $2N \times 2N$ unitary matrices A satisfying $A^tJA = J$, $J = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$.
COE	$U(N)/O(N)$, symmetric unitary $N \times N$ matrices identified with the above cosets via $B \to B^tB$.
CSE	$U(2N)/USp(2N)$, $2N \times 2N$ unitary matrices satisfying $J^tH^tJ = H$ identified by $B \to BJB^tJ^t$.

Table 1: Symmetric spaces $G(N)$, from [46].

Following the notation of Katz and Sarnak, we let $G(N)$ stand for any of the ensembles in Table 1. Since $A \in G(N)$ is a unitary matrix, we write the eigenvalues as $e^{i\theta_1(A)}, \ldots, e^{i\theta_N(A)}$, and label them so that $0 \le \theta_1(A) \le \cdots \le \theta_N(A) < 2\pi$. The measure, denoted dA, on the compact groups is always Haar measure, and for the circular ensembles we use the volume form as the probability measure.

Amongst other local statistics, Katz and Sarnak define the k-th *consecutive spacings*

$$\mu_k(A)[a,b] = \frac{\#\{1 \le j \le N | \frac{N}{2\pi}(\theta_{j+k} - \theta_j) \in [a,b]\}}{N}, \tag{24}$$

and show that for fixed $k \geq 1$, the same limit

$$\lim_{N\to\infty} \int_{G(N)} \mu_k(A)dA = \mu_k(CUE), \tag{25}$$

exists irrespective of how $G(N)$ is chosen from among the first four ensembles in Table 1. (Katz and Sarnak write $\mu_k(GUE)$, but since the local statistics of the ensemble of Hermitian matrices, GUE, and the CUE ensemble of unitary matrices are the same in the limit $N \to \infty$, either notation is suitable.) Further, Katz and Sarnak show that for a typical (in measure) $A \in G(N)$ the statistic $\mu_k(A)$ approaches $\mu_k(CUE)$ as $N \to \infty$. The same type of result is established also for the n-point correlations of the eigenvalues, and thus the local statistics of the entire set of N (or $2N$) eigenvalues of matrices from any of the four compact groups mentioned above tend to the same limit as N becomes large.

In contrast to this, Katz and Sarnak showed that the statistics of only the first eigenvalue (or more generally the first few eigenvalues) are specific to the particular ensemble chosen. If we define the distribution of the k-th eigenvalue of a matrix A varying over $G(N)$,

$$\nu_k(G(N))[a,b] = \text{meas}\{A \in G(N) : \tfrac{\theta_k(A)N}{2\pi} \in [a,b]\}, \tag{26}$$

then Katz and Sarnak show that the limit

$$\lim_{N\to\infty} \nu_k(G(N)) = \nu_k(G) \tag{27}$$

exists, but in contrast to (25), the limit depends on the ensemble G.

Based on this, Katz and Sarnak proposed that while the high zeros of any one L-function will always show the typical statistics of $U(N)$ (otherwise known as the CUE) the statistics of the lowest zeros near to $s = 1/2$ will show, when their distribution is determined over a suitable family of L-functions, the statistics of $U(N)$, $O(N)$ or $USp(2N)$.

To be more explicit, if we assume the Riemann hypothesis for the L-functions we are considering, then we write the zeros as $1/2 + i\gamma^{(n)}$ and order them

$$\cdots \leq \gamma^{(-2)} \leq \gamma^{(-1)} \leq 0 \leq \gamma^{(1)} \leq \gamma^{(2)} \leq \cdots . \tag{28}$$

If $\mathcal{F}$ denotes a family of L-functions, and an individual L-function within the family is identified by f and has conductor c_f, the zeros near $s = 1/2$ are normalized to have unit mean spacing by scaling them in the following way:

$$\frac{\gamma_f^j \log c_f}{2\pi}. \tag{29}$$

Let $\mathcal{F}_X$ denote the members of the family $\mathcal{F}$ with conductor less than X. Then Katz and Sarnak define the distribution of the jth eigenvalue as

$$\nu_j(X,\mathcal{F})[a,b] = \frac{\#\left\{f \in \mathcal{F}_X : \frac{\gamma_f^{(j)} \log c_f}{2\pi} \in [a,b]\right\}}{\#\mathcal{F}_X}. \tag{30}$$

It is then expected, and Katz and Sarnak provide analytical and numerical evidence for this, that $\nu_j(X,\mathcal{F})$ will converge, as X grows large, to $\nu_j(G(N))$, where $G(N)$ represents the symmetry type of the family: $U(N)$, $O(N)$ or $USp(2N)$. Similarly, any other statistics of the lowest zeros would also be expected, upon averaging over the family, to tend to the random matrix statistics of the correct symmetry type in the same limit as above.

For example, in the case of the family of L-functions with real Dirichlet characters described at (22), the low-lying zeros appear to show symplectic symmetry. In Figure 2, taken from [60], we see for $j = 1$ and $j = 2$ the good agreement between the numerically calculated distribution of the jth zero above $s = 1/2$ on the critical line and the distribution of the jth eigenvalue of the group $USp(2N)$. These distributions are visibly different from those of the lowest eigenvalues of matrices from $U(N)$ where there is no repulsion of the first zero by $\theta = 0$.

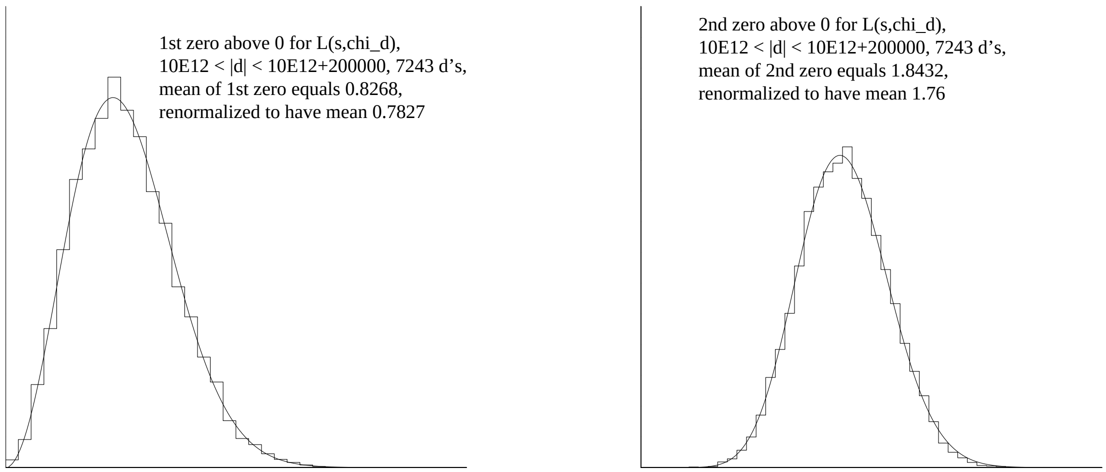

Figure 2: The numerically computed histogram of the distribution over d of the height of the first zero above $s = \frac{1}{2}$ of $L(s, \chi_d)$ (left) versus $\nu_1(Sp)$ and the distribution of the height of the second zero above $s = \frac{1}{2}$ versus $\nu_2(Sp)$ (right). Figure courtesy of M. O. Rubinstein [60].

The question of determining the symmetry type of a given family a priori is in general a difficult one. The method used by Katz and Sarnak is that for some families of L-functions, a related family of zeta functions on finite fields can be defined. In the case of these zeta functions the definition of families is straightforward, the Riemann hypothesis has been proven (in that all zeros lie on a circle) and the symmetry type is determined by the monodromy of the family (see [45]). The symmetry type of the related family of L-functions is then assumed to be the same. We return to this problem of determining the symmetry type of families in a later section.

Further studies of the statistics of low-lying zeros of Dirichlet L-functions have recently been carried out by Hughes and Rudnick [41], who compared the moments of linear statistics of scaled zeros around the symmetry point $s = \frac{1}{2}$ with similar moments of statistics of eigenphases near the point 1. In both cases they found comparable mock-Gaussian behaviour.

The idea of relating zero statistics for L-functions to averages over the classical compact groups has been extended by Keating, Linden and Rudnick [50] to the exceptional Lie groups. Specifically, they construct a family of L-functions associated with a finite field in which the relevant average is over the exceptional group G_2.

5 Random matrices and $\log \zeta(1/2 + it)$

In the preceding sections we have described the overwhelming evidence for a connection between the statistics of the zeros of L-functions and the eigenvalues of ensembles of matrices. The next step is to see what use can be made of that connection. Since the value distribution of a function is, to some extent, determined by its zeros, in this section and the following one we will describe how the value distributions of L-functions and of logarithms of L-functions can be probed using random matrix theory. We start with the logarithm.

At a given height t up the critical line, $\log\zeta(1/2+it)$ is a complex number, and one might ask: how are the real and imaginary parts of it distributed as t varies? In the limit as $t\to\infty$, the answer to this question is provided by a beautiful theorem due to Selberg [63, 59]

Theorem 4 (Selberg)

For any rectangle $B\in\mathbb{C}$,

$$\lim_{T\to\infty}\frac{1}{T}\left|\left\{t : T\le t\le 2T, \frac{\log\zeta(1/2+it)}{\sqrt{\frac{1}{2}\log\log T}}\in B\right\}\right|$$
$$= \frac{1}{2\pi}\int\int_B e^{-(x^2+y^2)/2}dx\,dy.$$

That is, in the limit as T, the height up the critical line, tends to infinity, the value distributions of the real and imaginary parts of $\log\zeta(1/2+iT)/\sqrt{(1/2)\log\log T}$ each tend, independently, to a Gaussian with unit variance and zero mean. Crucially for us, Odlyzko's computations for these distributions when $T\approx t_{10^{20}}$ show significant systematic deviations from this limiting form [59]. For example, increasing moments of both the real and imaginary parts diverge markedly from the Gaussian values. There is, of course, no contradiction; this merely suggests that the limiting Gaussian distribution is approached rather slowly as $T\to\infty$. It does, though, lead to the question of how to model the statistical properties of $\log\zeta(1/2+it)$ when t is large but finite.

Given its success in describing other statistical properties of the zeta function, it is natural ask whether random matrix theory might be used as the basis of such a model. The question is, then: what property of a matrix plays the role of the zeta function? The answer is simple: since the zeros of the zeta function are distributed like the eigenvalues of a random unitary matrix, the zeta function might be expected to be similar, in respect of its value distribution, to the function whose zeros are the eigenvalues, that is, to the *characteristic polynomial* of such a matrix. This idea was introduced and investigated in detail in [52]. Here we give details of some main results.

The characteristic polynomial of a unitary matrix A may be defined by

$$\Lambda(z)\equiv\Lambda_A(z)=\det(I-Az). \tag{31}$$

The moment generating function for $\mathrm{Re}\log\Lambda(e^{i\theta})$, for example, is thus

$$M_U(s;N)=\int_{U(N)}\exp(s\mathrm{Re}\log\Lambda_A(e^{i\theta}))dA=\int_{U(N)}|\Lambda_A(e^{i\theta})|^s dA, \tag{32}$$

where the integration is, as before, with respect to Haar measure. Obviously Λ_A may be written in terms of the eigenangles of A:

$$\Lambda_A(e^{i\theta})=\prod_{n=1}^{N}\left(1-e^{i(\theta_n+\theta)}\right). \tag{33}$$

Haar measure on $U(N)$ may also be expressed in terms of these eigenangles [64], allowing one to write

$$\int_{U(N)}|\Lambda_A(e^{i\theta})|^s dA=\frac{1}{(2\pi)^N N!}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{1\le j<m\le N}\left|e^{i\theta_j}-e^{i\theta_m}\right|^2\left|\prod_{n=1}^{N}(1-e^{i(\theta_n+\theta)})\right|^s d\theta_1\cdots d\theta_N. \tag{34}$$

This N-dimensional integral may then be computed by relating it to an integral evaluated by Selberg [54], giving

$$M_U(s;N) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+s)}{(\Gamma(j+s/2))^2}. \tag{35}$$

All information about the value distribution of $\text{Re}\log\Lambda$ on the unit circle is contained within (35): moments may be computed in terms of the derivatives of $M_U(s;N)$ at $s=0$, and the value distribution itself is the fourier transform of $M_U(iy;N)$. In the same way, information about the value distribution of $\text{Im}\log\Lambda$, and the joint value distribution of the real and imaginary parts of $\log\Lambda$ may be computed. This leads to a central limit theorem for $\log\Lambda$ (see also [3, 28]):

Theorem 5 (Keating and Snaith 2000 [52])

$$\lim_{N\to\infty} \text{meas}\left\{ A \in U(N) : \frac{\log\Lambda}{\sqrt{\frac{1}{2}\log N}} \in B \right\} = \frac{1}{2\pi} \int\int_B e^{-\frac{1}{2}(x^2+y^2)} dxdy$$

for rectangles $B \in \mathbb{C}$.

This theorem corresponds precisely to Selberg's for the value distribution of $\log\zeta(1/2+it)$, suggesting that random matrix theory, in the limit as the matrix-size tends to infinity, can indeed model the value distribution of $\log\zeta(1/2+it)$ as $t\to\infty$. The question that remains is whether it can also model the approach to the limit, that is, the value distribution when t is large but finite.

In order to relate the large-t asymptotics for the zeta function to the large-N asymptotics for the characteristic polynomials we need a connection between t and N. Note that the scaling in Theorem 4 and that in Theorem 5 coincide if we set

$$N = \log\frac{t}{2\pi}. \tag{36}$$

Such an identification is natural, because it corresponds to equating the mean density of the Riemann zeros at height t to the mean density of eigenphases for $N\times N$ unitary matrices, and these are the only parameters that appear in the connection between the respective statistics (cf. (3) and (9)). This therefore prompts the question as to whether the rate of approach to Selberg's theorem as $t\to\infty$ is related to that for Theorem 5 as $N\to\infty$ (which can be computed straightforwardly using (35)) if we make the identification (36).

As already noted above, Odlyzko's numerical computations of the value distribution of the zeta function near to the 10^{20}th zero show significant deviations from the Gaussian limit given in Theorem 4. The integer closest to $\log(t_{10^{20}}/2\pi)$ is $N=42$ ($t_{10^{20}} \approx 1.5202\times10^{19}$), so in Figure 3 we plot the value distribution for $\text{Re}\log\zeta(1/2+it)$, scaled as in Theorem 4, computed by Odlyzko [59], together with the value distribution for $\text{Re}\log\Lambda$, scaled as in Theorem 5, with respect to matrices taken from $U(42)$. Also shown is the Gaussian with zero mean and unit variance which represents the limit distribution in both cases (as $t\to\infty$ and $N\to\infty$ respectively). The negative logarithm of these curves is plotted in Figure 4, highlighting the behaviour in the tails. In order to quantify the data, the moments of the three distributions are listed in Table 2.

It is clear that random matrix theory provides an accurate description of the value distribution of $\text{Re}\log\zeta(1/2+it)$. It also models $\text{Im}\log\zeta(1/2+it)$ equally well [52]. This then suggests that, statistically, the zeta function at a large height t up the critical line behaves like a polynomial of degree N, where t and N are related by (36); and, moreover, that the polynomial in question is the characteristic polynomial of a random unitary matrix.

Figure 3: The value distribution for $\mathrm{Re}\log\Lambda$ with respect to matrices taken from $U(42)$, Odlyzko's data for the value distribution of $\mathrm{Re}\log\zeta(1/2+it)$ near the 10^{20}th zero (taken from [59]), and the standard Gaussian, all scaled to have unit variance. (Taken from [52].)

Moment	ζ a)	ζ b)	$U(42)$	Normal
1	0.0	0.0	0.0	0
2	1.0	1.0	1.0	1
3	-0.53625	-0.55069	-0.56544	0
4	3.9233	3.9647	3.89354	3
5	-7.6238	-7.8839	-7.76965	0
6	38.434	39.393	38.0233	15
7	-144.78	-148.77	-145.043	0
8	758.57	765.54	758.036	105
9	-4002.5	-3934.7	-4086.92	0
10	24060.5	22722.9	25347.77	945

Table 2: Moments of $\mathrm{Re}\log\zeta(1/2+it)$, calculated by Odlyzko over two ranges (labelled a and b) near the 10^{20}th zero ($t \simeq 1.520\times 10^{19}$) (taken from [59]), compared with the moments of $\mathrm{Re}\log\Lambda$ for $U(42)$ and the Gaussian (normal) moments, all scaled to have unit variance.

Figure 4: minus the logarithm of the value distributions plotted in Figure 3. (Taken from [52].)

Of course, specific properties of the zeta function would be expected to appear in the description of its value distribution. The point is that these contribute at lower order in the asymptotics, with the leading order being given by random matrix theory. For example, it is shown in [52] that as $N \to \infty$

$$\int_{U(N)} (\mathrm{Im} \log \Lambda_A)^2 dA = \frac{1}{2} \log N + \frac{1}{2}(\gamma + 1) + o(1), \tag{37}$$

where γ is Euler's constant, while Goldston [34] has proved, under the assumption of the Riemann Hypothesis and Montgomery's conjecture, that as $T \to \infty$

$$\begin{aligned} &\frac{1}{T} \int_0^T (\mathrm{Im} \log \zeta(1/2 + it))^2 dt \\ &= \frac{1}{2} \log \log \frac{T}{2\pi} + \frac{1}{2}(\gamma + 1) + \sum_{m=2}^{\infty} \sum_p \frac{(1-m)}{m^2} \frac{1}{p^m} + o(1). \end{aligned} \tag{38}$$

These expressions coincide under the identification (36), except for the sum over primes in (38). Obviously the primes have their origin in number theory, rather than random matrix theory.

In determining the value distribution of $\log \Lambda_A(e^{i\theta})$ (e.g. as in Theorem 5), the averages were performed over matrices A taken uniformly with respect to Haar measure on the unitary group $U(N)$. It is natural to ask how close this average is to an average with respect to θ when A is fixed; that is, about *ergodicity*. It was proved in [40] that indeed the average is ergodic, in the sense that in the limit as $N \to \infty$, the average over θ equals that over A for all but a set of matrices of zero measure.

As has been described above, the scaling of $\log \Lambda_A$ with respect to $\frac{1}{2} \log N$ leads to a central limit theorem. What about different scalings, characterizing, for example, the large deviations of

$\log \Lambda_A$? These were also computed in [40], and shown to agree with numerical calculations (e.g. the behaviour seen in the tails in Figure 4) and other results known to hold for the zeta function.

6 Random matrices and moments of L-functions

Determining the value distribution of an L-function is, it turns out, a significantly harder problem than determining the value distribution of its logarithm. Selberg's theorem completely characterizes the limiting distribution of $\log \zeta(1/2 + it)$, while for $\zeta(1/2 + it)$ almost nothing is known.

Regarding the moments of $|\zeta(1/2 + it)|$, there is a long-standing and important conjecture that $f(\lambda)$, defined by

$$\lim_{T\to\infty} \frac{1}{\log^{\lambda^2} T} \frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2\lambda} dt = f(\lambda)a(\lambda), \tag{39}$$

where

$$a(\lambda) = \prod_p \left\{ (1 - 1/p)^{\lambda^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(\lambda + m)}{m!\Gamma(\lambda)} \right)^2 p^{-m} \right) \right\}, \tag{40}$$

exists, and a much-studied problem then to determine the values it takes, in particular for integer λ (see, for example, [63, 43]). Obviously $f(0) = 1$. In 1918, Hardy and Littlewood proved that $f(1) = 1$ [36], and in 1926 Ingham proved that $f(2) = 1/12$ [42]. No other values are known. Based on number-theoretical arguments, Conrey and Ghosh have conjectured that $f(3) = 42/9!$ [24], and Conrey and Gonek that $f(4) = 24024/16!$ [25].

Given the success of random matrix theory in describing the value distribution of $\log \zeta(1/2+it)$, it is natural to ask whether it has anything to contribute on this issue. Invoking the identification (36), the question for the characteristic polynomials that is analogous to (39) is whether

$$f_U(\lambda) = \lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \int_{U(N)} |\Lambda_A(e^{i\theta})|^{2\lambda} dA \tag{41}$$

exists, and, if it does, what values it takes. The answer to this question was given in [52], where the following theorem is proved.

Theorem 6 (Keating and Snaith 2000 [52])

The coefficient f_U, defined as in (41), exists and is given by

$$f_U(\lambda) = \frac{G^2(1 + \lambda)}{G(1 + 2\lambda)},$$

where G denotes the Barnes G-function [5]. Hence $f_{CUE}(0) = 1$ (trivial) and

$$f_U(k) = \prod_{j=0}^{k-1} \frac{j!}{(j + k)!}$$

for integers $k \geq 1$.

Thus, for example, $f_U(1) = 1$, $f_U(2) = 1/12$, $f_U(3) = 42/9!$ and $f_U(4) = 24024/16!$. The fact that these values coincide with those associated, or believe to be associated, with the zeta function strongly suggests that

Conjecture 2 (Keating and Snaith 2000 [52])

With $f(\lambda)$ and $f_U(\lambda)$ defined as in (39) and (41), respectively,

$$f(\lambda) = f_U(\lambda)$$

for all $\mathrm{Re}\lambda > -1/2$.

This conjecture is also supported by Odlyzko's numerical data for non-integer values of λ between zero and two [52].

All that has been said so far on the moments of the Riemann zeta function deals only with the leading order coefficient, which survives the limit in (39). In recent work of Conrey, Farmer, Keating, Rubinstein and Snaith [23], however, conjectures arrived at through heuristic number theoretic arguments produce in a concise form all the significant lower-order terms, and are remarkably similar to the corresponding results for random matrix characteristic polynomials. Specifically, the $2k$th moment of the zeta function and the $2k$th moment of Λ_A are polynomials of degree k^2. In the random matrix case, this polynomial can be written down explicitly using (35). In the zeta function case, we do not know the analogue of (35). The coefficients in the polynomial can instead be computed from the expression conjectured below. We show then that (35) can be written in a form that is analogous to this expression. The expressions in question are written in terms of contour integrals (in the manner suggested by [19]) and involve the Vandermonde:

$$\Delta(z_1,\ldots,z_m) = \prod_{1\le i<j\le m}(z_j - z_i). \tag{42}$$

Conjecture 3 (Conrey, Farmer, Keating, Rubinstein and Snaith 2002 [23])

$$\begin{aligned}&\int_0^T \zeta(\tfrac{1}{2}+it+\alpha_1)\cdots\zeta(\tfrac{1}{2}+it+\alpha_k)\zeta(\tfrac{1}{2}-it-\alpha_{k+1})\cdots\zeta(\tfrac{1}{2}-it-\alpha_{2k})dt\\ &= \int_0^T W_k(\log\tfrac{t}{2\pi};\alpha_1,\ldots,\alpha_k;\alpha_{k+1},\ldots,\alpha_{2k})(1+O(t^{-\frac{1}{2}+\epsilon}))dt,\end{aligned}$$

where

$$\begin{aligned}&W_k(x;\alpha_1,\ldots,\alpha_k;\alpha_{k+1},\ldots,\alpha_{2k})\\ &= e^{\frac{x}{2}(-\alpha_1-\alpha_2-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_{2k})}\frac{(-1)^k}{k!^2(2\pi i)^{2k}}\oint\cdots\oint e^{\frac{x}{2}\sum_{j=1}^k z_j - z_{j+k}}\\ &\quad\times\frac{G(z_1,\ldots,z_{2k})\Delta^2(z_1,\ldots,z_{2k})}{\prod_{i=1}^{2k}\prod_{j=1}^{2k}(z_i-\alpha_j)}\,dz_1\ldots dz_{2k},\end{aligned}$$

with the path of integration being small circles surrounding the poles α_i. Here

$$G(z_1,\ldots,z_{2k}) = A_k(z_1,\ldots,z_{2k})\prod_{i=1}^k\prod_{j=1}^k\zeta(1+z_i-z_{j+k}),$$

and A_k is the Euler product

$$A_k(z) = \prod_p\prod_{i=1}^k\prod_{j=1}^k\left(1-\frac{1}{p^{1+z_i-z_{j+k}}}\right)\int_0^1\prod_{j=1}^k\left(1-\frac{e(\theta)}{p^{1/2+z_j}}\right)^{-1}\left(1-\frac{e(-\theta)}{p^{1/2-z_{j+k}}}\right)^{-1}\,d\theta,$$

with $e(\theta)=\exp(2\pi i\theta)$.

This is to be compared with the following theorem in random matrix theory:

Theorem 7 (Conrey, Farmer, Keating, Rubinstein and Snaith 2002 [22])

$$\begin{aligned}
&W_k(U(N);\alpha_1,\ldots,\alpha_{2k})\\
&=\int_{U(N)} \Lambda_A(e^{-\alpha_1})\cdots\Lambda_A(e^{-\alpha_k})\Lambda_{A^\dagger}(e^{\alpha_{k+1}})\cdots\Lambda_{A^\dagger}(e^{\alpha_{2k}})dA\\
&= e^{\frac{N}{2}(-\alpha_1-\alpha_2-\cdots-\alpha_k+\alpha_{k+1}+\cdots\alpha_{2k})}\frac{(-1)^k}{(2\pi i)^{2k}k!^2}\oint\cdots\oint e^{\frac{N}{2}\sum_{j=1}^k z_j - z_{j+k}}\\
&\qquad\times\prod_{\substack{1\le\ell\le k\\ k+1\le q\le 2k}}(1-e^{z_q-z_\ell})^{-1}\times\frac{\Delta^2(z_1,\ldots,z_{2k})}{\prod_{i=1}^{2k}\prod_{j=1}^{2k}(z_i-\alpha_j)}dz_1\cdots dz_{2k}.
\end{aligned}$$

The autocorrelation function in Theorem 7 is defined so as to be comparable with that in Conjecture 3, since in the case that α_i is purely imaginary, $e^{\mp\alpha_i}$ sits on the unit circle, in analogy with $1/2+it\pm\alpha_i$ lying on the critical line when α_i is purely imaginary in the Riemann zeta version. With these definitions the structures of $W_k(U(N);\alpha_1,\ldots,\alpha_{2k})$ and $W_k(x;\alpha_1,\ldots,\alpha_{2k})$ are identical if $\prod_{\ell=1}^k\prod_{m=1}^k(1-e^{z_{m+k}-z_\ell})^{-1}$ plays the role of $G(z_1,\ldots,z_{2k})$: crucially, these functions have poles in the same places. $G(z_1,\ldots,z_{2k})$, however, contains arithmetic information about the zeta function which clearly cannot be predicted by random matrix theory. The multiple integral expressions for $W_k(U(N);\alpha_1,\ldots,\alpha_{2k})$ and $W_k(x;\alpha_1,\ldots,\alpha_{2k})$ resemble those obtained by Brézin and Hikami [19] for the autocorrelation functions of the characteristic polynomials of Hermitian matrices in the limit of large matrix size N. In addition, expressions which are exact for finite N have been derived – by Mehta and Normand [55] in the case of moments and (in this issue) Fyodorov and Strahov [32] for autocorrelation functions – for quantities involving characteristic polynomials of matrices from Hermitian ensembles. For the ensembles of unitary matrices autocorrelations of characteristic polynomials have been calculated independently, using Lie theory, by Nonnenmacher and Zirnbauer [58].

That $W_k(x;0,0,\ldots,0)$ is actually a polynomial of degree k^2 can be seen by considering the order of the pole at $z_j=0$. From the numerator of the integrand we extract the coefficient of $\prod z_i^{2k-1}$, a polynomial of degree $2k(2k-1)$. The Vandermonde determinant squared is a homogeneous polynomial of degree $2k(2k-1)$. However, the poles coming from the $\zeta(1+z_i-z_{j+k})$ cancel k^2 of the Vandermonde factors. This requires us, in computing the residue, to take, in the Taylor expansion of $\exp(\frac{x}{2}\sum_1^k z_j-z_{j+k})$, terms up to degree k^2.

The fact that $W_k(\log\frac{t}{2\pi};0,0,\ldots,0)$ is a polynomial in $\log\frac{t}{2\pi}$ of degree k^2 corresponds nicely to $W_k(U(N);0,\ldots,0)=\int_{U(N)}|\Lambda_A(1)|^{2k}dA$, which is a polynomial of degree k^2 in N, as can be seen from (35). As was already mentioned, equating the density of the Riemann zeros at height t with the density of the random matrix eigenvalues yields the equivalence $N=\log\frac{t}{2\pi}$. After some manipulation one can see that a_k in (40) equals $A_k(0,\ldots,0)$ and so the leading term of the integral expression in Conjecture 3 for $W_k(x;0,\ldots,0)$ coincides precisely with the leading term conjectured by Keating and Snaith described above. The expression in Conjecture 3 has recently been shown to be connected with the analytic properties of multiple Dirichlet series [29].

Thus it appears that random matrix theory, specifically results concerning the characteristic polynomials of random unitary matrices, leads to a conjectural solution, supported by all available evidence, to the long-standing problem of calculating the moments of the Riemann zeta function on its critical line. In a similar way, the ideas of Katz and Sarnak, detailed in Section 4, suggest that averages over families of L-functions of the value at the symmetry point $s=1/2$ should be predicted by matrix ensemble averages over $U(N)$, $O(2N)$ or $USp(2N)$ (depending on the symmetry type of the family of L-functions) of the characteristic polynomial evaluated at the point 1 (the symmetry point of the eigenvalues). The results of Conrey and Farmer [21] and Keating and Snaith [51] show that the evidence that the leading order coefficient of such mean values splits into a product of a random matrix coefficient (calculated in the same way as $f_U(\lambda)$, but for the average over the appropriate group) and a product over primes specific to the family under consideration is as strong as that for the moments of the Riemann zeta function high on the critical line described earlier in this section. For example, for the family of L-functions mentioned at (22) the moment is conjectured, for large D, to be

Conjecture 4 (Conrey and Farmer 2000 [21], Keating and Snaith 2000 [51])

$$\frac{\pi^2}{6D} \sum_{|d|\le D}^{*} L(1/2,\chi_d)^k \sim f(k)a(k)(\tfrac{1}{2}\log D)^{k(k+1)/2},$$

where $\sum^*$ *is over fundamental discriminants,* $\chi_d(n) = (\frac{d}{n})$ *is the Kronecker symbol, and the sum is over all real, primitive Dirichlet characters of conductor up to* D*. Here*

$$a(k) = \prod_p \frac{(1-\frac{1}{p})^{\frac{k(k+1)}{2}}}{1+\frac{1}{p}} \left(\frac{(1-\frac{1}{\sqrt{p}})^{-k} + (1+\frac{1}{\sqrt{p}})^{-k}}{2} + \frac{1}{p} \right)$$

and the coefficient $f(k)$ *is*

$$\begin{aligned} f(k) &= \lim_{N\to\infty} \frac{1}{N^{k(k+1)/2}} \int_{USp(2N)} |\Lambda_A(1)|^k dA \\ &= \lim_{N\to\infty} \frac{1}{N^{k(k+1)/2}} 2^{2Nk} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\Gamma(\frac{1}{2}+k+j)}{\Gamma(\frac{1}{2}+j)\Gamma(1+k+N+j)}. \end{aligned}$$

For integer k*, this is*

$$f(k) = 2^{k(k+1)/2} \prod_{j=1}^{k} \frac{j!}{(2j)!}.$$

This conjecture agrees with previous results for $k = 1, 2, 3$, and the case of $k = 4$ is almost within reach of current methods [33, 44, 62]. A similar conjecture exists for families of L-functions with orthogonal symmetry.

Further, in [23] the same similarities of structure as we saw in Conjecture 3 and Theorem 7 are found between values of L-functions averaged over families and autocorrelations of random matrix characteristic polynomials. L-functions satisfy the functional equation

$$L(s) = \varepsilon X_L(s)\overline{L}(1-s), \tag{43}$$

where $X_L(s)$ is a Γ-factor similar to that appearing in (2) which is specific to the L-function. ε has modulus one and appears to provide another method of determining the symmetry type of a given family. We note that the characteristic polynomial

$$\Lambda_A(s) = \det(I - As) \tag{44}$$

of the unitary matrix A satisfies the functional equation

$$\Lambda_A(s) = \det A(-s)^N \overline{\Lambda_A}(\tfrac{1}{s}). \tag{45}$$

Here the transformation $s \to \frac{1}{s}$ plays the same role relative to the zeros of $\Lambda(s)$ on the unit circle as $s \to 1-s$ plays relative to the zeros of $L(s)$ on the line $\mathrm{Re}s = \frac{1}{2}$. Similarly, ε takes a role analogous to $\det A$. If A is drawn at random from $U(N)$ (with Haar measure) then $\det A$ can take any complex value with modulus one. However, as A varies over $O(2N)$, the determinant can only take the values ± 1, and for A in $USp(2N)$ we always have $\det A = +1$. In analogy to this, it seems that when ε varies over a family of L-functions and takes complex values with $|\varepsilon| = 1$, then the low-lying zeros of that family display eigenvalue statistics of the unitary group. If within a family ε takes the value both $+1$ and -1, then the zeros near $s = \frac{1}{2}$ are expected to behave like the eigenvalues of matrices from $O(2N)$, while values $\varepsilon = 1$ throughout the family would imply symplectic symmetry.

It is convenient, however, to define

$$Z_L(s) := \varepsilon^{-\frac{1}{2}} X_L^{-\frac{1}{2}}(s) L(s), \tag{46}$$

which satisfies the functional equation

$$Z_L(s) = \overline{Z_L}(1-s), \tag{47}$$

as well as the analogous characteristic polynomial

$$\mathcal{Z}_A(s) = e^{-i\phi/2}(-s)^{-N/2}\Lambda_A(s). \tag{48}$$

The functional equation

$$\mathcal{Z}_A(s) = \overline{\mathcal{Z}_A}(\tfrac{1}{s}) \tag{49}$$

is then the precise analogue of (47).

For the family $L(1/2, \chi_d)$ determined by the real Dirichlet characters (see (22)), averages of Z_L over the conductor $d < D$ are conjectured to have the form [23]:

Conjecture 5 (Conrey, Farmer, Keating, Rubinstein and Snaith 2002 [23])

Suppose $g(u)$ is a suitable weight function. Then, if $\mathcal{F}$ is the family of real Dirichlet L-functions with fundamental discriminants $d < 0$ we have

$$\sum_{L\in\mathcal{F}} Z_L(\tfrac{1}{2}+\alpha_1)\cdots Z_L(\tfrac{1}{2}+\alpha_k) g(d) = \sideset{}{^*}\sum_{d<0} Q_k(\alpha, \log\tfrac{|d|}{2\pi}) g(|d|)(1 + O(|d|^{-\frac{1}{2}+\epsilon})),$$

in which

$$\begin{aligned} Q_k(\alpha, x) \quad &= \quad \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k} \\ &\times \oint\cdots\oint \frac{G_-(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{\ell=1}^k\prod_{j=1}^k (z_j-\alpha_\ell)(z_j+\alpha_\ell)} e^{\frac{x}{2}\sum_{j=1}^k z_j}\, dz_1\ldots dz_k, \end{aligned} \tag{50}$$

where the path of integration encloses the $\pm\alpha$'s. Here

$$G_-(z_1,\ldots,z_k) = A_k(z_1,\ldots,z_k)\prod_{j=1}^k \left(\frac{\Gamma(\frac{3}{4}+\frac{z_j}{2})2^{z_j}}{\Gamma(\frac{3}{4}-\frac{z_j}{2})}\right)^{\frac{1}{2}} \prod_{1\le i\le j\le k}\zeta(1+z_i+z_j),$$

and A_k is the Euler product, which is absolutely convergent for $|\mathrm{Re} z_j| < 1/2$, for $j = 1, \ldots, k$, defined by

$$\begin{aligned} A_k(z_1,\ldots,z_k) &= \prod_p \prod_{1\le i\le j\le k}\left(1-\frac{1}{p^{1+z_i+z_j}}\right) \\ &\times \left(\frac{1}{2}\left(\prod_{j=1}^{k}\left(1-\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}+\prod_{j=1}^{k}\left(1+\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}\right)+\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^{-1}. \end{aligned}$$

There is a similar conjecture for the analogous sum over positive fundamental discriminants. For this conjecture G_- is replaced by G_+, where

$$G_+(z_1,\ldots,z_k) = A_k(z_1,\ldots,z_k)\prod_{j=1}^{k}\left(\frac{\Gamma(\frac{1}{4}+\frac{z_j}{2})2^{z_j}}{\Gamma(\frac{1}{4}-\frac{z_j}{2})}\right)^{\frac{1}{2}}\prod_{1\le i\le j\le k}\zeta(1+z_i+z_j),$$

and A_k is as before.

Since the symmetry type of this family is believed to be symplectic, we compare the mean value above with the following theorem [22]:

Theorem 8 (Conrey,Farmer,Keating,Rubinstein and Snaith 2002 [22])
We have the following integral over the unitary symplectic group with Haar measure:

$$\begin{aligned} \int_{USp(2N)} \mathcal{Z}_A(e^{-\alpha_1})\cdots\mathcal{Z}_A(e^{-\alpha_k})dA &= \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k} \\ \times \oint\cdots\oint \prod_{1\le \ell\le m\le k}(1-e^{-z_m-z_\ell})^{-1} &\frac{\Delta(z_1^2,\ldots,z_k^2)^2\prod_{j=1}^{k}z_j}{\prod_{i=1}^{k}\prod_{j=1}^{k}(z_j-\alpha_i)(z_j+\alpha_i)}e^{N\sum_{j=1}^{k}z_j}dz_1\cdots dz_k, \end{aligned}$$

where the contours of integration enclose the $\pm\alpha$'s.

We notice once again that the structure of the multiple integral in Theorem 8 is the same as that in (50), the difference only arising in the replacement of $\prod_{1\le\ell\le m\le k}(1-e^{-z_m-z_\ell})^{-1}$ by $G_\pm(z_1,\ldots,z_k)$; these two alternatives having the same set of poles as a result of the product over zeta functions in G. Equating the density of the L-function zeros near the point $s = 1/2$ and the density of the eigenvalues on the unit circle results in the equivalence $N = \frac{1}{2}\log\frac{|d|}{2\pi}$. The result in Conjecture 5 agrees with the Conjecture 4 for the leading order term. Similar results to Theorem 8 and Conjecture 5 are presented in [23] for families displaying orthogonal symmetry. Thus these results on the one hand lend support to the supposition that low-lying zeros of L-functions in families follow random matrix statistics in the manner proposed by Katz and Sarnak, while on the other illustrate the uses of random matrix theory in answering difficult number theoretical questions.

Conjecturing the value distribution of L-functions at $s = 1/2$ via random matrix theory using the techniques described above ties in to other important questions in number theory; for instance, for a family associated with elliptic curves, the number of L-functions which vanish at $s = 1/2$ is connected to the Birch and Swinnerton-Dyer Conjecture. This number can be predicted using random matrix theory [26]. The basic idea is that the L-functions in question form an orthogonal family and so their value distribution at $\frac{1}{2}$ can be written down using the analogue of Conjecture 4. It has also been shown that random matrix theory proves equally successful in the study of the derivative of the Riemann zeta function [38, 39, 56]. It is unlikely that the uses of random matrix theory in number theory end with the applications discussed in this review.

7 Final Remarks

The obvious question one is left with is: what is the reason for the connection between random matrices and L-functions? It has long been imagined there might be a spectral interpretation of the zeros. If the Generalized Riemann Hypothesis is true, such an interpretation could be the reason why; for example, if the zeros t_n of $\zeta(s)$ are the eigenvalues of a self-adjoint operator, or the eigenphases of a unitary operator, then automatically they would all be real. Some speculations along these lines are reviewed in [11], others have been pursued by Connes and co-workers. If the zeros are indeed related to the eigenvalues of a self-adjoint or unitary operator, and if that operator behaves 'typically', this would then suggest that the zeros might be distributed like the eigenvalues of random matrices. Alternatively, the success of random matrix theory in describing properties of the zeta function might be interpreted as evidence in favour of a spectral interpretation.

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